Msc Sem-03, Paper-11

Reflexive spaces. (Important theorems).

Definition:— The mapping of L into L^{**} defined by the linear transformation J is called the canonical mapping of L into L^{**}.

Def:— If range of J i.e. $R(J) = L^{**}$, then the linear space L is said to be algebraically reflexive.

Theorem:— Every finite dimensional linear space is algebraically reflexive.

Proof:— Let L be a finite dimensional linear space. We know that the mapping J from L into L^{**} is a linear transformation from L into L^{**} and J is one-one. Since L is finite dimensional, therefore $\dim L = L^* = \dim L^{**}$. Hence J is one-one implies J must be onto. therefore $R(J) = L^{**}$ and thus L is algebraically reflexive.

Theorem:— If L is an infinite-dimensional linear space, then it is not algebraically reflexive. Consequently a linear space is algebraically reflexive if and only if it is finite dimensional.

Proof:— Let $B = \{x_i : i \in I\}$ be a Hamel basis for L. Since L is finite dimensional, therefore the index set I is an infinite set and $x_i \neq x_j$ if $i \neq j$. Define $f_i \in L^*$ by $f_i(x_i) = 1$, $f_i(x_j) = 0$ if $i \neq j$. We claim that the set $\{f_i : i \in I\}$ is a linearly independent subset of L^*.

For suppose that $\{f_{i1}, f_{i2}, \ldots, f_{in}\}$ is any finite subset of the set $\{f_i : i \in I\}$.

Let $\alpha_1, \ldots, \alpha_n$ be scalars such that

$\alpha_1 f_{i1} + \ldots + \alpha_n f_{in} = \hat{0}$ (origin of L^*)

$\Rightarrow (\alpha_1 f_{i1} + \ldots + \alpha_n f_{in})(x) = \hat{0}(x) \; \forall x \in L$

$\Rightarrow \alpha_1 f_{i1}(x) + \ldots + \alpha_n f_{in}(x) = 0 \; \forall x \in L$

$\Rightarrow \alpha_1 f_{i1}(x_{i\mu}) + \ldots + \alpha_n f_{in}(x_{i\mu}) = 0, \; \mu = 1, \ldots, n$

Putting $x = x_{i\mu}$, where $\mu = 1, \ldots, n$:

$\Rightarrow \alpha_\mu = 0, \; \mu = 1, 2, \ldots, n.$

Thus $\{f_i : i \in I\}$ is a linearly independent subset of L^*. So it can be extended to form a basis for L^*. Let B^* be a Hamel basis of L^* which contains the set $\{f_i : i \in I\}$.

Let $\{\beta_i : i \in I\}$ be a set of scalars such that $\beta_i \neq 0$ for infinitely many indices i. Define $f^* \in L^{**}$ by setting $f^*(f_i) = \beta_i$ and $f^*(f) = 0$ if $f \in B^*$ but f is not one of the elements f_i. We shall show that f^* is not in the range of J i.e. there exists no element $x \in L$ such that $J(x) = f^*$.

Let $x \in L$ be such that $J(x) = f^*$.

Then by def. of J, we have $f^* = f_x^*$. Therefore $f^*(f_i) = f_x^*(f_i) = f_i(x) = \alpha_i$, where α_i is the coefficient of x_i in the representation of x in terms of the Hamel basis B. Now $\alpha_i = 0$ for all but a finite number of indices i. Therefore $f^*(f_i) = 0$ for all but a finite number of indices i. But according to our def. of f^*, we have $f^*(f_i) \neq 0$ for infinitely many indices i. Thus we get a contradiction. Hence there exists no $x \in L$ such that $J(x) = f^*$. Therefore the mapping J is not onto L^{**}. i.e. range of $J \neq L^{**}$.

Hence L is not algebraically reflexive.

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